EXACTLY SOLVABLE TWO-DIMENSIONAL COMPLEX MODEL WITH REAL SPECTRUM

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Supersymmetrical intertwining relations of second order in derivatives allow to construct a two-dimensional quantum model with complex potential, for which *all* energy levels and bound state wave functions are obtained analytically. This model *is not amenable* to separation of variables, and it can be considered as a specific complexified version of generalized two-dimensional Morse model with additional $\sinh^{-2}$ term. The energy spectrum of the model is proved to be purely real. To our knowledge, this is a rather rare example of a nontrivial exactly solvable model in two dimensions. The symmetry operator is found, the biorthogonal basis is described, and the pseudo-Hermiticity of the model is demonstrated. The obtained wave functions are found to be common eigenfunctions both of the Hamiltonian and of the symmetry operator. This paper is dedicated to the eightieth birthday of Yuri Victorovich Novozhilov.

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1. Introduction

While the role of exactly solvable and partially (quasi-exactly) solvable models in one-dimensional Quantum Mechanics is well known[d], the number of these models is rather restricted. For two-dimensional systems (exact- and partial-) solvability is a much harder task. Besides models with separation of variables, which are actually reduced to couples of one-dimensional models, only Calogero and the so called Calogero-like models[1] (with the number of particles $N = 3$)[e] are exactly solvable. Some partially solvable systems were constructed in [3], [4], [5], [6] by the supersymmetrical method of $SUSY-$separation variables. Just the methods of supersymmetry seem to be the most appropriate ones to attack this fundamental problem of constructing exactly and partially solvable quantum (and classical [7]) models in the case of few space dimensions.

The first step in this direction was taken in [8], where a list of partial solutions of supersymmetrical intertwining relations for two-dimensional systems was found. All these models are integrable, i.e. there is a dynamical symmetry of fourth order in momenta, such that the corresponding generators are in involution with the Hamiltonians. After that two new supersymmetrical methods - of $SUSY$-separation of variables and of shape invariance - were proposed [3], [5], [6], [9] which provided an opportunity to obtain part of the spectrum and wave functions for some of these integrable models. Thus a class of partially solvable integrable two-dimensional quantum systems was built. Naturally, it would be very important to obtain some *exactly* solvable models among these integrable ones.

In the present paper a new idea, which may be useful in order to solve the task formulated above, will be presented. Namely, we will choose a particular value of parameter ($a = -1/2$) in the two-dimensional Morse potential with additional $\sinh^{-2}$ term [3] in such a way that one

[d]Here and below we call the system exactly solvable if all eigenvalues and eigenfunctions of its bound states are known analytically. The system is called partially (quasi-exactly) solvable when a part of them is known.

[e]Incidentally, we would like to mention that like the Calogero models all considered models (including this generalized Morse model) cam alternatively be interpreted [2] either as describing three particles on a line or as a particle in a two-dimensional space.

partner Hamiltonian in the SUSY intertwining relations *will allow* for standard separation of variables. In such a case this Hamiltonian is exactly solvable, and its spectrum and eigenfunctions will be found analytically. The second partner Hamiltonian still *does not allow for* the separation of variables, but due to intertwining relations its spectrum and eigenfunctions can also be obtained analytically by acting with the supercharge. Since SUSY intertwining relates the spectra of partner Hamiltonians up to normalizable zero modes of supercharges, the requirement of exact solvability of both Hamiltonians amounts to the condition that all zero modes of supercharges are under control. This could be implemented for the generalized two-dimensional Morse model (with additional $\sinh^{-2}$ term), but as will be shown in Section 2, the direct analysis of singularities of wave functions for $a = -1/2$ case is rather problematic in a general form due to the presence of confluent hypergeometric functions. Nevertheless the principal opportunities offered by the method are demonstrated for a particular energy level. Consideration in Section 3 of some specific complex version of this model (which regularizes the repulsive singularity for the real values of x_1, x_2) leads to a more interesting system: *all* bound states and normalizable wave functions can be derived explicitly. The properties of this last model are investigated. The most important feature is the reality of the energy spectrum (which does not depend on the parameter of complexification δ). In addition, in Section 4 the bound-state-biorthogonal-basis [10] will be constructed in a natural way, and the action of symmetry operator will be calculated. In particular, it will be shown that the obtained wave functions are the common eigenfunctions both of the Hamiltonian and of the symmetry operators.

2. Generalized two-dimensional Morse potential with $a = -1/2$

The main element of the two-dimensional second order Supersymmetrical Quantum Mechanics (SUSY QM) [8], [5] is represented in terms of the intertwining relations:

$$\tilde{H}Q^+ = Q^+H; \quad Q^-\tilde{H} = HQ^-; \tag{1}$$

between a pair of scalar Hamiltonians $H,\, \tilde{H}$:

$$H = -\triangle + V(\vec{x}); \quad \tilde{H} = -\triangle + \tilde{V}(\vec{x}); \quad \triangle \equiv \partial_1^2 + \partial_2^2; \quad \partial_i \equiv \partial/\partial x_i; \quad i = 1, 2. \tag{2}$$

These intertwining relations realize the isospectrality up to zero modes of $Q^{\pm}$ of the superpartners $H,\, \tilde{H}$ and the connection between their wave functions with the same values of energy:

$$\Psi_n(x) = Q^-\tilde{\Psi}_n(x); \quad \tilde{\Psi}_n(x) = Q^+\Psi_n(x); \quad n = 0, 1, 2, ... \tag{3}$$

Though the solutions of intertwining relations (1) have to be searched in principle for the supercharges $Q^{\pm}$ with the most general form of second order differential operators (see [8], [5] for details), we will restrict ourselves to the particular solution of Lorentz (hyperbolic) type:

$$Q^+ = (\partial_1^2 - \partial_2^2) + C_i\partial_i + B = 4\partial_+\partial_- + C_+\partial_- + C_-\partial_+ + B; \tag{4}$$

$$Q^- = (\partial_1^2 - \partial_2^2) - C_i\partial_i + B = 4\partial_+\partial_- - C_+\partial_- - C_-\partial_+ + B; \; i = 1, 2. \tag{5}$$

Then the potentials $\tilde{V}(\vec{x}), V(\vec{x})$ and the function $B(\vec{x})$ can be expressed in terms of four functions - $F_1(2x_1)$, $F_2(2x_2)$ and $C_{\pm}(x_{\pm})$:

$$\begin{aligned} \tilde{V} &= \frac{1}{2}(C_+' + C_-') + \frac{1}{8}(C_+^2 + C_-^2) + \frac{1}{4}\Big(F_2(x_+ - x_-) - F_1(x_+ + x_-)\Big), \\ V &= -\frac{1}{2}(C_+' + C_-') + \frac{1}{8}(C_+^2 + C_-^2) + \frac{1}{4}\Big(F_2(x_+ - x_-) - F_1(x_+ + x_-)\Big), \end{aligned} \tag{6}$$

$$B = \frac{1}{4}\Big(C_+C_- + F_1(x_+ + x_-) + F_2(x_+ - x_-)\Big). \tag{7}$$

These functions must satisfy the following equation:

$$\partial_-(C_-F) = -\partial_+(C_+F),$$

where $x_{\pm} \equiv x_1 \pm x_2 \quad \partial_{\pm} = \partial/\partial x_{\pm}$ and $C_{\pm}$ depend only on $x_{\pm}$, respectively:

$$C_+ \equiv C_1 - C_2 \equiv C_+(x_+); \quad C_- \equiv C_1 + C_2 \equiv C_-(x_-),$$

and

$$F = F_1(x_+ + x_-) + F_2(x_+ - x_-).$$

Among different particular solutions of this class [8], [5] we will choose here the well studied two-dimensional generalization of one dimensional Morse potential. The model is not amenable to standard separation of variables, and it is defined by:

$$\begin{aligned}
C_+ &= 4a\alpha; \quad C_- = 4a\alpha\cdot\coth\frac{\alpha x_-}{2} && (8)\\
f_1(x_1) &\equiv \frac{1}{4}F_1(2x_1) = -A\Big(\exp(-2\alpha x_1) - 2\exp(-\alpha x_1)\Big); && (9)\\
f_2(x_2) &\equiv \frac{1}{4}F_2(2x_2) = +A\Big(\exp(-2\alpha x_2) - 2\exp(-\alpha x_2)\Big) && (10)\\
\tilde{V}(\vec{x}) &= \alpha^2 a(2a-1)\sinh^{-2}\Big(\frac{\alpha x_-}{2}\Big) + 4a^2\alpha^2 + \\
&+ A\Big[\exp(-2\alpha x_1) - 2\exp(-\alpha x_1) + \exp(-2\alpha x_2) - 2\exp(-\alpha x_2)\Big] && (11)\\
V(\vec{x}) &= \alpha^2 a(2a+1)\sinh^{-2}\Big(\frac{\alpha x_-}{2}\Big) + 4a^2\alpha^2 + \\
&+ A\Big[\exp(-2\alpha x_1) - 2\exp(-\alpha x_1) + \exp(-2\alpha x_2) - 2\exp(-\alpha x_2)\Big], && (12)
\end{aligned}$$

where parameters a, $A > 0$, $\alpha > 0$ are arbitrary real numbers.

Just on the basis of this model two new methods - $SUSY$-separation of variables and two-dimensional shape invariance - were elaborated for the first time [3], [5]. As a result, partial solvability of the model was discovered [3], [5], [9] and a variety of wave functions $\Psi_n, \tilde{\Psi}_n$ was found analitycally for parameters restricted to the range:

$$a \in (-\infty, -\frac{1}{4} - \frac{1}{4\sqrt{2}}); \quad \frac{\sqrt{A}}{\alpha} - n - \frac{1}{2} > -2a > 0; \quad n = 0, 1, 2, ..., \qquad (13)$$

which provides the condition of normalizability of zero modes of Q^+ and the absence of the "fall to the centre" (see details in [3],[5]).

It is worth to recall now the main idea of the method of $SUSY$-separation of variables. By similarity transformation the supercharge $Q^{\pm}$ (4) for Lorentz metrics can be transformed to the operators without linear derivatives:

$$\begin{aligned}
q^{\pm} &= \exp(-\chi(\vec{x}))Q^{\pm}\exp(+\chi(\vec{x})) = \partial_1^2 - \partial_2^2 + \frac{1}{4}(F_1(2x_1) + F_2(2x_2)); && (14)\\
\chi(\vec{x}) &\equiv -\frac{1}{4}\Big(\int C_+(x_+)dx_+ + \int C_-(x_-)dx_-\Big),
\end{aligned}$$

which allow for the separation of variables (this is why we call this method as *SUSY*-separation). If $-F_1$ and $+F_2$ belong to a class of exactly solvable one-dimensional potentials, the normalizable zero modes $\Omega_n(\vec{x})$ of Q^+ can be found analytically. Then due to (1) a set of wave functions of H can be constructed as linear combinations of these zero modes, leading therefore to partial solvability of the model. The shape invariance method [3] (together with the second shape invariance [9] of the model) enlarges the part of the spectrum known analytically.

Now a new method, involving separation of variables in one partner Hamiltonian (which does not hold for the other partner), will be formulated. Let us choose the parameters of the same model in such a way that the Hamiltonian H **does allow** for the standard procedure of separation of variables. Then the intertwining relations and knowledge of zero modes of supercharges provides full information about the partner Hamiltonian $\tilde{H}$, which **does not allow** conventional separation of variables.

Luckily, a suitable choice of the value of the parameter $a = -1/2$ in (12) makes H amenable to separation of variables:

$$H(\vec{x}) = h_1(x_1) + h_2(x_2) + \alpha^2; \quad h_1(x_1) \equiv -\partial_1^2 - f_1(x_1); \quad h_2(x_2) \equiv -\partial_2^2 + f_2(x_2), \tag{15}$$

and its wave functions with energies

$$E_{n,m} = \epsilon_n + \epsilon_m + \alpha^2, \tag{16}$$

where the last term originates from the free term in (12), can be written as:

$$\Psi_{E_{n,m}} = c_1\eta_n(x_1)\eta_m(x_2) + c_2\eta_m(x_1)\eta_n(x_2); \quad c_1, c_2 = Const \tag{17}$$

where ϵ_k and $\eta_k(x)$ solve the exactly solvable one-dimensional problem with the standard Morse potential in terms of confluent hypergeometric functions (see [11]):

$$\Big(-\partial^2 + A\Big(\exp(-2\alpha x) - 2\exp(-\alpha x)\Big)\Big)\eta_k(x) = \epsilon_k\eta_k(x) \tag{18}$$

$$\eta_k = \exp(-\frac{\xi}{2})(\xi)^{s_k}\Phi(-k, 2s_k + 1; \xi); \quad \xi \equiv \frac{2\sqrt{A}}{\alpha}\exp(-\alpha x); \tag{19}$$

$$\epsilon_k = -A[1 - \frac{\alpha}{\sqrt{A}}(k + 1/2)]^2; \quad s_k = \frac{\sqrt{A}}{\alpha} - k - 1/2. \tag{20}$$

Thus the two-dimensional Schrödinger problem with Hamiltonian H is obviously exactly solvable with 2-fold degeneracy for the levels with $n \neq m$.

Now the next step of our approach is to use the intertwining relations (1) in order to obtain, by means of (3), the wave functions of the Hamiltonian $\tilde{H}$, which *does not allow* for separation of variables. It is clear from the explicit form of Q^+ that due to the singularity of Q^+ at $x_- \to 0$ the normalizability of functions $\tilde{\Psi}$ should be investigated in detail. Though this problem is not analyzable fully due to the presence of hypergeometric functions in (17), particular normalizable eigenfunctions of $\tilde{H}$ can be studied. Indeed, taking antisymmetric ($c_1 = -c_2$) wave functions $\Psi^A_{E_{n,m}}$ of the form (17), one can check[f] that for $|n-m| = 1$ they are represented as linear combinations of zero modes $\Omega_k(\vec{x})$ of Q^+:

$$\Psi^A_{E_{m+1,m}} = -\Psi^A_{E_{m,m+1}} = \frac{2(s_{m+1}+1)(2s_{m+1}+1)}{2s_{m+1}+m+2}\Sigma_{k=0}^{k=m} a_{m+1,k}\Omega_k(\vec{x}),$$

and therefore have *no partner* bound states at $\tilde{H}$. The symmetric functions ($c_1 = c_2$) from (17) with $n = m \pm 1$ give $Q^+\Psi^S_{E_{m\pm1,m}}$, which are also absent among bound states of $\tilde{H}$ due to their nonnormalizable behaviour at $x_- \to 0$. Therefore the spectrum of $\tilde{H}$ definitely *does not* include $E_{m\pm1,m}$.

The wave function $\Psi^A_{E_{0,2}}$ leads to the (symmetrical) $\tilde{\Psi}^S_{E_{0,2}}$, which can be rewritten as:

$$\tilde{\Psi}^S_{E_{0,2}} \equiv Q^+\Psi^A_{E_{0,2}} = \alpha^2(2s_2+3)(\xi_1-\xi_2)^2\exp(-\frac{\xi_1+\xi_2}{2})(\xi_1\xi_2)^{s_2}.$$

This example demonstrates explicitly that a suitable choice of c_1, c_2, n, m at (17) can compensate the singularity in Q^+ and provide a normalizable wave function $\tilde{\Psi}_{E_{n,m}}$.

In principle, besides the eigenfunctions $\tilde{\Psi}$ of $\tilde{H}$ discussed above, some additional normalizable eigenfunctions could exist. If so, due to intertwining relations the action of the operator Q^- onto these functions should either give zeros or unnormalizable functions Ψ. The first option is excluded by the analysis of zero modes of $Q^-(a=-1/2) \equiv Q^+(a=+1/2)$, taking into account that $a = +1/2$ is outside the range (13). The second option also does

[f] It is necessary to use elementary relations between confluent hypergeometric functions (see [11], vol.1, Subsection 6.3.) and some properties derived in Subsection 4.4. of [3].

not materialize since normalizable eigenfunction of $\tilde{H}$ with potential (11) for $a = -1/2$ gives $\tilde{\Psi} \sim x_-^2$ at $x_- \to 0$. Therefore action of Q^- cannot affect its normalizability, and we are back to the case of previous paragraph.

3. Complexified model

Since the bound state spectrum of the model $\tilde{H}$ for $a = -1/2$ turned out to be difficult to analyze due to singularities of $\tilde{V}(\vec{x})$ and $Q^\pm$ at $x_- = 0$, a natural idea is to remove somehow these singularities. In one-dimensional Quantum Mechanics the recipe is well known [12], [13], [14] - to shift the space coordinate into the complex plane. It means that from this moment we deal with complex potentials[g], in general.

In our two-dimensional situation[h] we have to shift $x_- = x_1 - x_2$, therefore one has to violate the exchange symmetry of the system under $x_1 \leftrightarrow x_2$. The easiest way is to replace

$$\vec{x} \to \vec{x} + i\vec{\delta}; \quad \vec{\delta} = (\delta, 0)$$

(with δ small enough, such that $\alpha\delta \in (0, \pi/2)$) removing the singularities from the real (x_1, x_2) plane. In terms of ξ of Eq.(19) a phase factor appears: $\xi \to e^{-i\alpha\delta}\xi$. As usual, such imaginary shift of $\vec{x}$ preserves **the reality of the spectrum** of the Schrödinger operator.

Under this shift not too many changes affect formulas of Section 2. The operator $Q^-(\vec{x} + i\vec{\delta})$ in (5) preserves its form, but from now on it is not hermitian conjugate of $Q^+((\vec{x} + i\vec{\delta}))$, namely $Q^- = \Big((Q^+)^\dagger\Big)^\star = (Q^+)^t$. Functions (8) - (10) become complex, and (also complex) potentials (11), (12) are related now by

$$\tilde{V}(\vec{x} + i\vec{\delta}) = \tilde{V}^\star(\vec{x} - i\vec{\delta}) = \exp(-2i\vec{\delta}\vec{\partial})\tilde{V}^\star(\vec{x} + i\vec{\delta})\exp(+2i\vec{\delta}\vec{\partial}).$$

This equality expresses the property of pseudo-Hermiticity for non-Hermitian Hamiltonians, which guarantees in general [10] that the spectrum consists of real eigenvalues and complex

[g] An extensive literature concerning one-dimensional non-Hermitian Hamiltonians followed the seminal papers [15] of C.Bender and S.Boettcher (see for example, [16]).

[h] Non-Hermitian models in two-dimensional Quantum Mechanics were studied in [4].

conjugated pairs. In particular, in our case the whole bound states spectrum of $\tilde{H}$ is known to be real.

The eigenfunctions $\eta_k(x)$ of one-dimensional Morse equation (17) are expressed in terms of confluent hypergeometric functions as in (19), but with $\xi \to e^{-i\alpha\delta}\xi$, and are still normalizable. One can check that *no additional normalizable* solution of (17) appears in the complex x-plane. Indeed, after the substitution $\eta_k \equiv \exp(-\xi/2)\xi^s Y(\xi)$, the Eq.(18) in the variable ξ is reduced to the confluent hypergeometrical equation [11]. There are different ways to represent the general solution of this equation. A convenient one is:

$$Y(\xi) = c_1 y_5(\xi) + c_2 y_7(\xi),$$

where ξ includes a phase factor, and definitions and terminology for linearly independent solutions y_5, y_7 are given in [11] (see Vol. 1, Subsection 6.7.):

$$y_5 = \Phi(a, b; \xi); \quad y_7 = \exp(\xi)\Psi(b - a, b; -\xi). \tag{21}$$

Just the exponential in the second term in (21) allows to prove that the only kind of *normalizable* solutions Y, even for our complex ξ, corresponds to $c_2 = 0$. Therefore, the conditions of normalizability lead to Eqs.(19), (20), and the whole bound state spectrum (16) of Hamiltonian $H(\vec{x} + i\vec{\delta})$ is known and it remains *real* after this complexification.

Using again the intertwining relations (1), one can obtain the eigenfunctions

$$\tilde{\Psi}_{E_{n,m}}(\vec{x} + i\vec{\delta}) = Q^+\eta_n(x_1 + i\delta)\eta_m(x_2).$$

Due to absence of singularity of Q^+ at $x_- \to 0$, these wave functions are normalizable.

One has to remember that the partner Hamiltonians $\tilde{H}$ and H are isospectral only up to zero modes of supercharges. Normalizable zero modes of Q^+ were constructed and investigated in detail in [3]. It was proved that a variety of linear combinations of these zero modes can be built, which are eigenfunctions of the Hamiltonian H. In particular, this can be done for the case $a = -1/2$ (when H allows separation of variables), and the corresponding eigenvalues are [3]:

$$E_k = 2\epsilon_k + 2\alpha^2 s_k = -2\alpha^2 s_k(s_k - 1) = \epsilon_{k+1} + \epsilon_k + \alpha^2; \quad s_k = \frac{\sqrt{A}}{\alpha} - k - 1/2. \tag{22}$$

For arbitrary n, m eigenvalues (16) of H are two-fold degenerate: there are symmetrical and antisymmetrical components in (17). But for the particular case of $n = m \pm 1$ *the antisymmetrical* combination $\Psi^A_{E_{m\pm1,m}}$ is annihilated by Q^+. In a contrast to the Hermitian model of Section 2, for *the symmetrical* component $\Psi^S_{E_{m\pm1,m}}$ its partner state $\tilde{\Psi}^A_{E_{m\pm1,m}} = Q^+\Psi^S_{E_{m\pm1,m}}$ for the complex model has no singularity at $x_- \to 0$, and therefore these energy levels $E_{m\pm1,m}$ exist in the spectrum of $\tilde{H}$, but they are not degenerate.

The possible zero modes of $Q^-(a) = Q^+(-a)$ can be also investigated by the same method as in [3], the corresponding eigenvalues are $E_k = -2\alpha^2 s_k + 2\epsilon_k$. Because these eigenvalues coincide with $E_{k,k-1}$ from the general expression (16), no new eigenstates can appear for $\tilde{H}$ due to zero modes of $Q^-(a)$.

Thus the complex model with the Hamiltonian $\tilde{H}(\vec{x} + i\delta)$ for $a = -1/2$ is **exactly solvable**, its spectrum is real:

$$E_{n,m} = \epsilon_n + \epsilon_m + \alpha^2. \tag{23}$$

For $n = m \pm 1$ these levels are not degenerate, but for all other n, m there is 2-fold degeneracy. The wave functions of $\tilde{H}$ are:

$$\tilde{\Psi}_{E_{n,m}} = Q^+\Psi_{E_{n,m}}, \tag{24}$$

where $\Psi_{E_{n,m}}$ are given by (17) with $\vec{x} \to \vec{x} + i\vec{\delta}; \quad \vec{\delta} = (\delta, 0)$.

4. Integrability and biorthogonality

It is known [8], [5] that "by construction" all Hamiltonians $\tilde{H}, H$, which are intertwined according to (1) by second order operators $Q^\pm$, are in involution with operators of fourth order in derivatives:

$$\tilde{R} = Q^+Q^-; \quad R = Q^-Q^+; \quad [H, R] = 0; \quad [\tilde{H}, \tilde{R}] = 0. \tag{25}$$

These symmetry operators[i] are not reducible to functions of Hamiltonians, and therefore all these systems, including the one presented in this paper, are **integrable**.

[i] Apart from the case of Laplacian metrics in supercharges, which is not considered here (see details in [8]).

In the particular case $a = -1/2$ with separation of variables in H, considered in this paper, the expressions for R and $\tilde{R}$ can be transformed by using the similarity transformation (14) and the specific form of one-dimensional Hamiltonians $h_1(x_1), h_2(x_2)$ in (15):

$$Q^{\pm} = \exp(\pm\chi) q^{\pm} \exp(\mp\chi) = \exp(\pm\chi)(h_2 - h_1)\exp(\mp\chi).$$

Then the symmetry operator R for the Hamiltonian with separation reads:

$$R = (h_2 - h_1)^2 + 2\alpha^2(h_1 + h_2) + \alpha^4,$$

and its wave functions (17) are simultaneously eigenfunctions of R with eigenvalues:

$$r_{n,m} = (\epsilon_m - \epsilon_n)^2 + 2\alpha^2(\epsilon_m + \epsilon_n) + \alpha^4.$$

The wave functions (24) of the Hamiltonian $\tilde{H}$ without separation of variables are also common eigenfunctions both of $\tilde{H}$ and $\tilde{R}$:

$$\tilde{R}\tilde{\Psi}_{E_{n,m}} = Q^+Q^-Q^+\Psi_{E_{n,m}} = r_{n,m}\tilde{\Psi}_{E_{n,m}}.$$

Thus the property, noticed in [3] for a few known wave functions, is fulfilled now for all wave functions in the case of the exactly solvable model, that we have constructed in this paper. Though almost all (for $n \neq m$) wave functions $\tilde{\Psi}_{E_{n,m}}$ are 2-fold degenerate, the symmetry operator $\tilde{R}$ does not mix these degenerate functions.

The factorized wave functions $\Psi_{E_{n,m}}(\vec{x} + i\vec{\delta})$ of $H(\vec{x} + i\vec{\delta})$ and their complex conjugate functions $\Psi^{\star}_{E_{n,m}}$ form the so called biorthogonal basis for the non-Hermitian Hamiltonian. The corresponding biorthogonality relations

$$\begin{aligned} < \Psi^{\star}_{E_{n,m}} \mid \Psi_{E_{n',m'}} >= \int d^2x \Psi_{E_{n,m}}(\vec{x} + i\vec{\delta}) \cdot \Psi_{E_{n',m'}}(\vec{x} + i\vec{\delta}) = \\ = \int dx_1 \eta_n(x_1 + i\delta) \cdot \eta_{n'}(x_1 + i\delta) \int dx_2 \eta_n(x_2) \cdot \eta_{n'}(x_2) = \delta_{nn'}\delta_{mm'} \end{aligned} \qquad (26)$$

can either be checked straightforwardly or by comparing this integral along the line $x_1 + i\delta$ with the analogous integral along the real x_1 for the case of $\delta = 0$ and real-valued wave functions η_n. Absence of singularities in the narrow strip between these lines means that

relations (26) follow from the standard orthogonality of wave functions for the Hermitian Hamiltonians. On the contrary, if one would consider the scalar product defined with the complex conjugation of the first multiplier in the integral in (26), the connection with the Hermitian case would be not so evident.

The bound-state-biorthogonal-basis for the non-Hermitian operator $\tilde{H}(\vec{x}+i\vec{\delta})$ consists of $\tilde{\Psi}_{E_{n,m}} = Q^+\Psi_{E_{n,m}}$ and $\tilde{\Psi}^{\star}_{E_{n,m}} = (Q^+)^{\star}\Psi^{\star}_{E_{n,m}}$. Due to the equality $Q^- = ((Q^+)^{\dagger})^{\star}$ the scalar products can be written as:

$$< \tilde{\Psi}^{\star}_{E_{n,m}} \mid \tilde{\Psi}_{E_{n',m'}} >=< (Q^+)^{\star}\Psi^{\star}_{E_{n,m}} \mid Q^+\Psi_{E_{n',m'}} >=< \Psi^{\star}_{E_{n,m}} \mid Q^-Q^+\Psi_{E_{n',m'}} > .$$

Since $\Psi_{E_{n',m'}}$ is an eigenfunction of the symmetry operator $R = Q^-Q^+$ with an eigenvalue $r_{n.m}$, biorthogonality for $\tilde{H}(\vec{x}+i\vec{\delta})$ follows directly from (26), thus leading to a diagonalization of $\tilde{H}$ in the bound state subspace.

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